

# A criterion for an abelian variety to be simple

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## Abstract

In this note we give a numerical criterion that expresses the condition that an abelian variety be simple in terms of an invariant that is closely related to the  $s$ -invariant of Ein-Cutkosky-Lazarsfeld.

## Introduction

An abelian variety is *simple* if it does not contain any non-trivial abelian subvarieties. The purpose of this note is to provide a criterion that expresses simpleness as a condition on the codimension one level:

**Theorem.** *Let  $(X, L)$  be a polarized abelian variety over the complex numbers. The following statements are equivalent:*

- (i) *For every line bundle  $M$  on  $X$  that is not proportional to  $L$  in  $\text{Num}(X) \otimes \mathbb{Q}$ , the supremum*

$$\sup \{ t \in \mathbb{R} \mid L - tM \text{ is nef} \}$$

*is irrational or equal to  $\infty$ .*

- (ii)  *$X$  is simple.*

Here  $\text{Num}(X)$  denotes the group of numerical equivalence classes of line bundles on  $X$ . A line bundle  $B$  is *nef* if  $B \cdot C \geq 0$  for every irreducible curve  $C$  in  $X$ .

The criterion provides new examples where the  $s$ -invariants of Cutkosky-Ein-Lazarsfeld are irrational (see Example 2.2). It may also be viewed as a statement about the geometry of the ample cone of  $X$  (see Remark 2.3).

*Conventions.* We work throughout over the field of complex numbers. Additive notation will be used for the tensor product of line bundles.

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# 1 Proof of the theorem

Let  $X$  be an abelian variety. For line bundles  $L$  and  $M$  on  $X$  we consider the number

$$\sigma(L, M) =_{\text{def}} \sup \{ t \in \mathbb{R} \mid L - tM \text{ is nef} \} \in \mathbb{R} \cup \{\pm\infty\} \quad (1)$$

We start by giving an algebraic characterization of  $\sigma(L, M)$ , when  $L$  is ample.

**Proposition 1.1** *Let  $(X, L)$  be a polarized abelian variety of dimension  $n$ , and let  $M$  be any line bundle on  $X$ . If  $\zeta(L, M)$  denotes the maximal root of the polynomial*

$$\chi(uL - M) = \frac{1}{n!} (uL - M)^n \in \mathbb{Q}[u]$$

*(all of whose roots are in any event real), then*

$$\sigma(L, M) = \begin{cases} \infty & \text{if } \zeta(L, M) \leq 0, \\ \frac{1}{\zeta(L, M)} & \text{if } \zeta(L, M) > 0. \end{cases}$$

*Proof.* We will make use of the isomorphism of  $\mathbb{Q}$ -vector spaces

$$\begin{aligned} \text{NS}_{\mathbb{Q}}(X) &\longrightarrow \text{End}_{\mathbb{Q}}^s(X) \\ B &\longmapsto f_B := \phi_L^{-1} \circ \phi_B, \end{aligned}$$

where  $\phi_L$  and  $\phi_B$  are the canonical homomorphisms  $X \rightarrow \hat{X}$  to the dual abelian variety  $\hat{X}$ . Note first that  $B$  is a nef class if and only if the  $\mathbb{Q}$ -endomorphism  $f_B$  (or, more precisely, the derivative  $d(mf_B) : T_0X \rightarrow T_0X$  of a suitable multiple  $mf_B$ ,  $m > 0$ ) has no negative eigenvalues. In fact, the characteristic polynomial  $P_B$  of  $f_B$  satisfies

$$P_B(m) = \frac{\chi(mL - B)}{\chi(L)}$$

for  $m \in \mathbb{Z}$  (see [5, Sect. 5.2]), so that its alternating coefficients are positive multiples of the intersection numbers

$$L^k B^{n-k} \quad (0 \leq k \leq n),$$

where  $n = \dim X$ . But  $B$  is nef if and only if all  $L^k B^{n-k}$  are non-negative (see e.g. [1, Lemma 1.1]), and – since  $f_B$  is symmetric and has therefore only real eigenvalues – this is equivalent to saying that  $P_B$  has no negative roots.

So in particular, for  $t \in \mathbb{Q}$  the line bundle  $L - tM$  is nef if and only if the endomorphism

$$f_{L-tM} = \text{id}_X - tf_M$$

has no negative eigenvalues. But the eigenvalues of  $\text{id}_X - tf_M$  are the numbers  $1 - t\lambda$  where  $\lambda$  runs through the set of eigenvalues of  $f_M$ . From this the assertion follows.  $\square$

*Proof of the theorem.* Suppose first that (i) holds and assume by way of contradiction that there is an abelian subvariety  $Y \subset X$  different from  $X$  and  $0$ . Consider the norm endomorphism  $N_Y \in \text{End}(X)$  of  $Y$  with respect to the polarization  $L$  (see [2] or [5, Sect. 5.3] for details on norm endomorphisms). The pullback

$$M =_{\text{def}} N_Y^* L$$

then corresponds to the endomorphism  $f_M = N_Y^2 = eN_Y$ , where  $e$  is the exponent of the induced polarization  $L|_Y$ , i.e., the minimal positive integer  $e$  such that

$$e\phi_{L|_Y}^{-1} \in \text{End}(X)$$

is an (integral) homomorphism. Since  $L$  is ample and  $M$  is non-trivial and nef, but certainly not ample, the bundles  $L$  and  $M$  are not proportional in  $\text{Num}(X) \otimes \mathbb{Q}$ . Now, the eigenvalues of  $N_Y$  are  $0$  and  $e$ , and hence Proposition 1.1 implies that

$$\sigma(L, M) = \frac{1}{e^2} \in \mathbb{Q} ,$$

which is a contradiction with (i).

Supposing now that (i) does not hold, we will show that (ii) does not hold as well. We can argue as in the first part of the proof of [1, Proposition 1.2]. In brief, if  $\sigma(L, M)$  is rational, then a suitable rational multiple of the class  $L - \sigma(L, M)M$  is an integral class  $B$  on  $X$ , which is nef but not ample. The homomorphism  $\phi_B$  has therefore a non-trivial kernel. On the other hand, since  $L$  and  $M$  are not proportional,  $B$  cannot be topologically trivial, and consequently  $\phi_B$  cannot be the zero morphism. So the connected component of its kernel containing the point  $0$  is a non-trivial abelian subvariety of  $X$ . This completes the proof.  $\square$

## 2 Complements and application to $s$ -invariants

We give here two further applications of Proposition 1.1, and we point out the geometric consequences of the theorem. Our first observation says in effect that if  $\sigma(L, M)$  is rational, then there are only finitely many possibilities for its value.

**Corollary 2.1** *Let  $(X, L)$  be a polarized abelian variety of dimension  $n$ , and let  $M$  be a line bundle on  $X$  such that  $-M$  is not nef. If  $\sigma(L, M)$  is a rational number, then it is of the form*

$$\sigma(L, M) = \frac{p}{q} ,$$

where  $p$  and  $q$  are coprime integers satisfying the divisibility conditions

$$p|L^n \quad \text{and} \quad q|M^n .$$

*Proof.* Write  $\sigma(L, M) = p/q$  with coprime integers  $p$  and  $q$ . By Proposition 1.1, the rational number  $q/p$  is the maximal root of the polynomial  $\chi(uL - M)$ . Now, the polynomial

$$n! \cdot \chi(uL - M)$$

has integer coefficients, its leading coefficient is  $L^n$ , and the constant term is (up to a possible sign)  $M^n$ . This implies the assertion.  $\square$

**Example 2.2 (Irrational  $s$ -invariants)** We establish here the relationship between our result and the  $s$ -invariants introduced by Cutkosky-Ein-Lazarsfeld in [3]. In particular we obtain many new examples of irrational  $s$ -invariants.

Consider a coherent ideal sheaf  $\mathcal{J} \subset \mathcal{O}_X$  on a smooth projective variety  $X$ , and let  $\nu : Y \rightarrow X$  be the blow-up of  $X$  along  $\mathcal{J}$ . We have  $\mathcal{J} \cdot \mathcal{O}_Y = \mathcal{O}_Y(-F)$  for an effective Cartier divisor  $F$  on  $Y$ . Fixing an ample divisor  $H$  on  $Y$ , the  $s$ -invariant of  $\mathcal{J}$  with respect to  $H$  is defined to be the positive real number

$$s_H(\mathcal{J}) = \min \{ s \in \mathbb{R} \mid s \cdot \nu^* H - F \text{ is nef} \}$$

(see [3, Sect. 1]). Interestingly, the  $s$ -invariant governs (among other things) the asymptotic regularity of powers of  $\mathcal{J}$  (see [3, Sect. 3]). When  $\mathcal{J}$  is the ideal sheaf of a point  $x \in X$ , the reciprocal of  $s_H(\mathcal{J})$  is the *Seshadri constant*  $\varepsilon(H, x)$ , as introduced by Demailly (see [4] and [6, Chapt. 5]). Paoletti ([8], [9]) has studied the case where  $\mathcal{J}$  is the ideal sheaf of a smooth curve in a threefold. It is natural to ask whether  $s$ -invariants can become irrational. While it is still unknown whether this can happen for Seshadri constants (i.e. when  $\mathcal{J}$  is the ideal sheaf of a point), it does happen for  $s$ -invariants in general. The first examples, due to Ein-Cutkosky-Lazarsfeld, are  $s$ -invariants of curves on suitable abelian surfaces (see [3, Example 1.7 and Example 1.11]). Our result clarifies the picture on abelian varieties in the following way. If  $(X, L)$  is a polarized abelian variety and  $D$  an effective divisor on  $X$ , then the number  $\sigma(L, D)$  defined in (1) is just the reciprocal of an  $s$ -invariant,

$$\sigma(L, D) = \frac{1}{s_L(\mathcal{J}_{D/X})},$$

where  $\mathcal{J}_{D/X}$  is the ideal sheaf of  $D$  in  $X$ . So the present result implies that on simple abelian varieties in fact *all* such  $s$ -invariants are irrational (as long as  $L$  and  $\mathcal{O}_X(D)$  are not numerically proportional), while on non-simple abelian varieties rational  $s$ -invariants occur.

**Remark 2.3** It may also be useful to think of the theorem – and the invariant  $\sigma(L, M)$  – in terms of the geometry of the ample cone as follows. Given a polarized abelian variety  $(X, L)$ , we may ask “how far”  $L$  is away from the boundary  $\partial \text{Amp}(X)$  of the ample cone of  $X$ . The number  $\sigma(L, M)$  is then just the distance of  $L$  to  $\partial \text{Amp}(X)$  in the direction of  $-M$ , measured in units of  $M$ . When  $X$  contains a non-trivial abelian subvariety  $Y$  and  $M = N_Y^* L$ , then the proof of the theorem tells us that this distance is  $\leq 1$ , and in fact a rational number of the form  $1/e^2$ .

Finally, we establish a lower bound on  $\sigma(L, M)$  that can be computed explicitly from the intersection numbers  $L^k M^{n-k}$ . In view of Example 2.2, this gives an upper bound on the corresponding  $s$ -invariant.

**Corollary 2.4** *Let  $(X, L)$  be a polarized abelian variety of dimension  $n$ , and let  $M$  be a line bundle on  $X$  such that  $-M$  is not nef. Then*

$$\sigma(L, M) \geq \left(1 + \max_{0 \leq k < n} \binom{n}{k} \frac{L^k M^{n-k}}{L^n}\right)^{-1}$$

*Proof.* By Riemann-Roch, the coefficient  $a_k$  at  $u^k$  in the polynomial  $\chi(uL - M)$  is given by

$$a_k = (-1)^{n-k} \frac{L^k M^{n-k}}{k!(n-k)!}$$

for  $0 \leq k \leq n$ . It is a theorem of Cauchy (see for instance [7, Theorem 27.2]) that all roots of a complex polynomial  $P(u) = \sum_{k=0}^n a_k u^k$ , with  $a_n \neq 0$ , lie within the circle around 0 of radius

$$1 + \max_{0 \leq k < n} \left| \frac{a_k}{a_n} \right|,$$

so that in our case we have in particular

$$\zeta(L, M) < 1 + \max_{0 \leq k < n} \binom{n}{k} \frac{L^k M^{n-k}}{L^n}$$

for the maximal root  $\zeta(L, M)$  of  $\chi(uL - M)$ . Since  $-M$  is not nef,  $\zeta(L, M)$  is positive, and hence Proposition 1.1 implies the assertion.  $\square$

When Proposition 1.1 is being used to bound  $\sigma(L, M)$ , the issue is to estimate the roots of polynomials in terms of their coefficients. The estimate used in the proof of Corollary 2.4 is among the most immediate ones; the reader may consult for instance [7] for more refined estimates.

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